1

Number and Arithmetic

In this chapter we review the key features of elementary numbers and arithmetic. The topics covered are those found to be most useful later on.

Prerequisites

It will be helpful if you know something about:

- simple types of numbers such as integers, fractions, negative numbers, decimals
- the concepts of 'greater than' and 'less than'
- elementary arithmetic: addition, subtraction, multiplication and division
- powers and indices notation, $2^3 = 2 \times 2 \times 2$, for example
- how to convert a simple fraction to a decimal and vice versa

Objectives

In this chapter you will find:

- different types of numbers and their properties (particularly zero)
- the use of inequality signs
- highest common factors and lowest common denominators
- manipulation of numbers (BODMAS)
- handling fractions
- factorial ($n!$) and combinatorial $\left({}^nC_r \text{ or } \binom{n}{r}\right)$ notation
- powers and indices
- decimal notation
- estimation of numerical expressions

Motivation

You may need the material of this chapter for:

- numerical manipulation and calculation in engineering applications
- checking and using scientific formulae
- illustrating and checking results used later in mathematics
- statistical calculations
- numerical estimation and 'back of an envelope' calculations

A note about calculators

Calculators obviously have their place, particularly in applied mathematics, numerical methods and statistics. However, they are very rarely needed in this chapter, and the skills it aims to develop are better learnt without them.

1.1 Review

1.1.1 Types of numbers

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A. For each number choose one or more descriptions from the following: (a) integer, (b) negative, (c) rational number (fraction), (d) real, (e) irrational, (f) decimal, (g) prime.
(i) is done as an example

(i) -1 (a, b, c, d)
(ii) $\frac{1}{2}$
(iii) 0
(iv) 7
(v) $\frac{23}{5}$
(vi) $-\frac{3}{4}$
(vii) 0.73
(viii) 11
(ix) 8
(x) $\sqrt{2}$
(xi) -0.49
(xii) π

B. Which of the following descriptions apply to the expressions in (i)–(x) below?

(a) infinite
(b) does not exist
(c) negative
(d) zero
(e) finite
(f) non-zero

(i) 0×1 (d, e)
(ii) $0 + 1$
(iii) $\frac{1}{0}$
(iv) $2 - 0$
(v) 0^2
(vi) $0 - 1$
(vii) $\frac{0}{0}$
(viii) $3 \times 0 + \frac{3}{0}$
(ix) $\frac{0^3}{0}$
(x) $\frac{2}{2}$

1.1.2 Use of inequality signs

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Express symbolically:

(i) x is a positive, non-zero, number ($x > 0$)
(ii) x lies strictly between 1 and 2
(iii) x lies strictly between -1 and 3
(iv) x is equal to or greater than -2 and is less than 2
(v) The absolute value of x is less than 2.

1.1.3 Highest common factor and lowest common multiple

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A. Express in terms of prime factors

(i) 15 ($= 3 \times 5$) (ii) 21 (iii) 60
(iv) 121 (v) 405 (vi) 1024
(vii) 221

B. Find the highest common factor (HCF) of each of the following sets of numbers

(i) 24, 30 (6) (ii) 27, 99 (iii) 28, 98
(iv) 12, 54, 78 (v) 3, 6, 15, 27

C. Find the lowest common multiple (LCM) of each of the following sets of numbers

(i) 3, 7 (21) (ii) 3, 9 (iii) 12, 18
(iv) 3, 5, 9 (v) 2, 4, 6

1.1.4 Manipulation of numbers

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Evaluate

(i) $2 + 3 - 7$ ($= -2$) (ii) $4 \times 3 \div 2$ (iii) $3 + 2 \times 5$
(iv) $(3 + 2) \times 5$ (v) $3 + (2 \times 5)$ (vi) $18 \div 2 \times 3$
(vii) $18 \div (2 \times 3)$ (viii) $-2 - (4 - 5)$ (ix) $(4 \div (-2)) \times 3 - 4$
(x) $(3 + 7) \div 5 + (7 - 3) \times (2 - 4)$

1.1.5 Handling fractions

➤12 28➤➤

A. Simplify

(i) $\frac{4}{6}\ \left(= \frac{2}{3}\right)$ (ii) $\frac{18}{9}$ (iii) $\frac{7}{3} \times \frac{4}{7}$
(iv) $\frac{7}{5} \times \frac{3}{14}$ (v) $\frac{3}{4} \div \frac{4}{5}$ (vi) $\frac{1}{2} + \frac{1}{3}$
(vii) $\frac{1}{2} - \frac{1}{3}$ (viii) $\frac{4}{15} - \frac{7}{3}$ (ix) $1 + \frac{1}{2} + \frac{1}{3}$
(x) $\frac{2}{3} - \frac{3}{4} + \frac{1}{8}$

B. If the numbers a and b are in the ratio $a : b = 3 : 2$ and $a = 6$, what is b?

1.1.6 Factorial and combinatorial notation – permutations and combinations

➤ 16 29 ➤➤

A. Evaluate

(i) $3!\ (=6)$ (ii) $6!$ (iii) $\dfrac{24!}{23!}$ (iv) $\dfrac{12!}{9!\,3!}$

B. (i) Evaluate (a) ${}^3C_2\ (=3)$ (b) ${}^6C_4$ (c) ${}^6P_3$

(ii) In how many ways can two distinct letters be chosen from ABCD?

(iii) How many permutations of the letters ABCDE are there?

1.1.7 Powers and indices

➤ 18 29 ➤➤

A. Reduce to simplest power form.

(i) $2^3 2^4\ (=2^7)$ (ii) $3^4/3^3$ (iii) $(5^2)^3$

(iv) $(3\times 4)^4/(9\times 2^3)$ (v) $16^2/4^4$ (vi) $(-6)^2\left(-\frac{3}{2}\right)^3$

(vii) $(-ab^2)^3/a^2b$ (viii) $2^2\left(\frac{1}{2}\right)^{-3}$

B. Express in terms of simple surds such as $\sqrt{2}$, $\sqrt{3}$, etc.

(i) $\sqrt{50}\ (=5\sqrt{2})$ (ii) $\sqrt{72}-\sqrt{8}$ (iii) $(\sqrt{27})^3$

(iv) $\left(\dfrac{\sqrt{2}\sqrt{3}}{4}\right)^2$ (v) $\dfrac{\sqrt{3}\sqrt{7}}{\sqrt{84}}$ (vi) $\dfrac{\sqrt{3}+2\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

(vii) $\left(\dfrac{3^{1/3}9^{1/3}}{27}\right)^2$

1.1.8 Decimal notation

➤ 22 30 ➤➤

A. Express in decimal form

(a) $\frac{1}{2}$ (b) $-\frac{3}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{7}$

B. Express as fractions

(a) 0.3 (b) 0.67 (c) $0.\dot{6}$ (d) 3.142

C. Write the following numbers in scientific notation, stating the mantissa and exponent.

(i) 11.00132 (ii) 1.56 (iii) 203.45 (iv) 0.0000321

D. Write the numbers in **C** to three significant figures.

1.1.9 Estimation

➤25 31➤➤

Estimate the approximate value of each of

(i) $\dfrac{4.5 \times 10^5 \times 2.0012}{8.892 \times 10^4}$ (ii) $\dfrac{\sqrt{254 \times 10^4 + 28764.5}}{2.01 \times 10 - 254 \times 10^{-6}}$

1.2 Revision

1.2.1 Types of numbers

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Numbers can be classified into different types:

- natural numbers
- zero
- directed numbers
- integers
- rational numbers (fractions)
- irrational numbers
- real numbers
- complex numbers (➤ Chapter 12).

The counting numbers

$$1, 2, 3, 4, \ldots$$

are called **natural numbers**.

Zero, 0, is really in a class of its own – we always have to be careful with it. It is an integer and also, of course, a real number. Essentially, zero enables us to define negative numbers. Thus, the negative of 3 is the number denoted $n = -3$ satisfying

$$3 + n = 0$$

This enables us to ‘count in opposite directions’ using **directed** or **negative numbers**

$$-1, -2, -3, -4, \ldots$$

The full set of numbers

$$\{\ldots -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

is called the set of **integers**.

Numbers that can be written in the form:

$$\frac{\text{integer}}{\text{non-zero integer}} \left(\text{e.g. } \frac{3}{4}, \frac{-1}{2}\right)$$

$$\left(\text{including integers, such as } 6 = \frac{6}{1}\right)$$

are called **rational numbers** or **fractions**. All measurements of a physical nature (length, time, voltage, etc.) can only be expressed in terms of such numbers. Numbers which are not rational, and cannot be expressed as ratios of integers, are called **irrational numbers**. Examples are $\sqrt{2}$ and π. We will prove that $\sqrt{2}$ is irrational in Chapter 14.

The set of all numbers: integers, rational and irrationals is called the set of **real numbers**. It can be shown that together these numbers can be used to 'label' every point on a continuous infinite line – the **real line**. So called 'complex numbers' are really equivalent to pairs of real numbers. They are studied in Chapter 12, and an introduction is provided in the Applications section of Chapter 2.

Note that zero, 0, is an exceptional number in that one cannot **divide** by it. It is not that 1/0 is 'infinity', but simply that **it does not exist at all**. **Infinity**, denoted ∞, is not really a number. It is a concept that indicates that no matter what positive (negative) number you choose, you can always find another positive (negative) number greater (less) than it. Crudely, ∞ denotes a 'number' that is as large as we wish.

Solution to review question 1.1.1

A. All numbers here are real, so d applies to them all.

(i) -1 is a negative natural number, i.e. an integer; (a, b, c, d)

(ii) $\frac{1}{2}$ is a ratio of integers and is therefore rational; (c, d)

(iii) 0 is an integer – the only one that is its own negative; (a, c, d)

(iv) 7 is a natural number and an integer. It is in fact also a prime number – that is, only divisible by itself or 1 (Section 1.2.3). It is also an **odd** number (cannot be exactly divided by 2); (a, c, d, g)

(v) $\dfrac{23}{5}$ is a rational number – actually an improper fraction (Section 1.2.5); (c, d)

(vi) $-\frac{3}{4}$ is a rational number – a proper fraction (Section 1.2.5); (b, c, d)

(vii) 0.73 is actually a decimal representation of a rational number

$$0.73 = \frac{73}{100}$$

sometimes called a decimal fraction, but usually simply a decimal (Section 1.2.8); (c, d, f)

(viii) 11 is a natural number and an integer – like 7 it is also prime, and is also odd as any prime greater than 2 must be; (a, c, d, g)

(ix) 8 is another natural number and integer – but it is not prime, since it can be written as $2 \times 2 \times 2 = 2^3$ (Section 1.2.7). It is also an even number; (a, c, d)

(x) The square root of 2, $\sqrt{2}$, is not a rational number. This can be shown by assuming that

$$\sqrt{2} = \frac{m}{n}$$

where m and n are two integers and deriving a contradiction. $\sqrt{2}$ is irrational and is a real number. Such numbers, square roots of prime numbers, are called **surds** (Section 1.2.7); (d, e)

(xi) -0.49 is a decimal representation (Section 1.2.8) of the negative rational number

$$-\frac{49}{100}$$

(b, c, d, f)

(xii) π, the ratio of the circumference of a circle to its diameter, is not a rational number – it is an **irrational number**. That is, it cannot be written as a fraction. 22/7, for example, is just an approximation to π; (d, e)

B. (i) $0 \times 1 = 0$, i.e. zero – which of course is also finite (d, e).

(ii) $0 + 1 = 1$, finite, non-zero (e, f).

(iii) $\frac{1}{0}$ does not exist – it is not infinite, negative, zero, finite or non-zero – it just does not exist (b).

(iv) $2 - 0 = 2$, finite and non-zero (e, f).

(v) $0^2 = 0 \times 0 = 0$, zero and finite (d, e).

(vi) $0 - 1 = -1$, negative, finite, non-zero (c, e, f).

(vii) $\frac{0}{0}$ does not exist (you can't 'cancel' the zeros!). It is not infinite, negative, zero, finite or non-zero – it just does not exist (b).

(viii) Because of the $\frac{3}{0}$ the expression $3 \times 0 + \frac{3}{0}$ does not exist (b).

(ix) $\dfrac{0^3}{0}$ again, does not exist (b).

(x) $\frac{2}{2} = 1$ – no problem here, finite and non-zero (e, f).

Note that none of the numbers in **B** is referred to as 'infinite'.

1.2.2 Use of inequality signs

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The real numbers are **ordered**. That is, we can always say whether one number a is less than, equal to, or greater than another given number b. To denote this we use the 'comparator' symbols or **inequalities**, $<$ and $\leq$, $>$ and $\geq$. $a > b$ means a is greater than b; $a < b$ means a is less than b. Thus $6 > 5$, $4 < 5$. $a \geq b$ means a is greater than or equal to b, and similarly $a \leq b$ means a is less than or equal to b. Be very careful to distinguish between, for example $a > b$ and $a \geq b$. Sometimes it is also useful to use the 'not equal to' symbol, $\neq$.

Care is needed when changing signs and forming reciprocals with inequalities. For example, if $a > b > 0$, then $-a < -b$ and $\dfrac{1}{a} < \dfrac{1}{b}$. However, if $a > 0 > b$ then $-a < -b$ is still true, but $\dfrac{1}{a} > \dfrac{1}{b}$. Try a few numerical examples to check these statements. Most of us find inequalities difficult to handle and they require a lot of practice. However, in this book we will need only the basic properties of inequalities. We will say more about inequalities in Section 3.2.6.

Often we wish to refer to the **positive** or **absolute** value of a number x (for example in a rectified sine wave). We denote this by the **modulus of x**, $|x|$. For example

$$|-4| = 4$$

By definition $|x|$ is never negative, so $|x| \geq 0$. Also, note that $|x| < a$ means $-a < x < a$. For example:

$$|x| < 3$$

means

$$-3 < x < 3$$

Solution to review question 1.1.2

(i) 'x is a positive non-zero number' is expressed by $x > 0$

(ii) 'x lies strictly between 1 and 2' is expressed by $1 < x < 2$

(iii) 'x lies strictly between -1 and 3' is expressed by $-1 < x < 3$

(iv) 'x is equal to or greater than -2 and is less than 2' is expressed by $-2 \leq x < 2$

(v) If the absolute value of x is less than 2 then this means that if x is positive then $0 \leq x < 2$, but if x is negative then we must have $-2 < x \leq 0$. So, combining these we must have $-2 < x < 2$. This can also be expressed in terms of the modulus as $|x| < 2$.

1.2.3 Highest common factor and lowest common multiple

◄3 28►

A **prime number** is a positive integer which cannot be expressed as a product of two or more smaller distinct positive integers. That is, a prime number cannot be divided exactly by any integer other than 1 or itself. From the definition, 1 is not a prime number. 6, for example, is not a prime, since it can be written as 2×3. The numbers 2 and 3 are called its (prime) **factors**. Another way of defining a prime number is to say that it is has no integer factors other than 1 and itself.

There are an infinite number of prime numbers:

$$2, 3, 5, 7, 11, 13, \ldots$$

but no formula for the nth prime has been discovered. Prime numbers are very important in the theory of codes and cryptography. They are also the 'building blocks' of numbers, since any given integer can be written uniquely as a product of primes:

$$12 = 2 \times 2 \times 3 = 2^2 3$$

This is called **factorising** the integer into its prime factors. It is an important operation, for example, in combining fractions.

The **highest common factor** (HCF) of a set of integers is the largest integer which is a factor of all numbers of the set. For small numbers we can find the HCF 'by inspection' – splitting the numbers into prime factors and constructing products of these primes that divide each number of the set, choosing the largest such product.

The **lowest common multiple** (LCM) of a set of integers is the smallest integer which is a multiple of all integers in the set. It can again be found by prime factorisation of the numbers. In this book you will only need to use the LCM in combining fractions and only for small, manageable numbers, so the LCM will usually be obvious 'by inspection'. In such cases one can normally guess the answer by looking at the prime factors of the numbers, and then check that each number divides the guess exactly.

Solution to review question 1.1.3

A. (i) $15 = 3 \times 5$

(ii) $21 = 3 \times 7$

(iii) $60 = 3 \times 20 = 3 \times 4 \times 5 = 2 \times 2 \times 3 \times 5 = 2^2 \times 3 \times 5$

(iv) $121 = 11 \times 11$

(v) $405 = 5 \times 81 = 5 \times 9 \times 9 = 5 \times 3 \times 3 \times 3 \times 3 = 3^4 \times 5$

(vi) $1024 = 4 \times 256 = 4 \times 16 \times 16$
$= 4 \times 4 \times 4 \times 4 \times 4 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$
$= 2^{10}$ (or, anticipating the rules of indices $= 4^5 = (2^2)^5 = 2^{10}$)

(vii) $221 = 13 \times 17$

Notice that there may be more than one way of factorising, but that the final result is always the same. You may also have noticed that it gets increasingly difficult to factorise, compared to multiplying – thus in (vii), it is so much easier to multiply 13×17 than to discover these factors from 221. This fact is actually the key idea behind many powerful coding systems – the **trap-door principle** – in some cases it is much easier doing a mathematical operation than undoing it!

B. (i) 24, 30. In this case it is clear that the largest integer that exactly divides these two is 6 and so the HCF of 24 and 30 is 6.

(ii) 27, 99. Again the fairly obvious answer here is 9.

(iii) 28, 98. Perhaps not so obvious, so split each into prime factors:

$$28 = 4 \times 7 = 2 \times 2 \times 7$$

$$98 = 2 \times 49 = 2 \times 7 \times 7$$

from which we see that the HCF is $2 \times 7 = 14$.

(iv) 12, 54, 78. Splitting into prime factors will again give the answer here, but notice a short cut: 2 clearly divides them all, leaving 6, 27, 39. 3 divides all of these leaving 2, 9, 13. These clearly have no factors in common (except 1) and so we are done, and the HCF is $2 \times 3 = 6$.

(v) 3, 6, 15, 27. Here 3 is the only number that divides them all and is the HCF in this case.

C. (i) Since 3, 7 are both primes, the LCM is simply their product, 21.

(ii) $9 = 3 \times 3$, so 3 and 9 both divide 9 and there is no smaller number that does so. The LCM is thus 9.

(iii) Since $3 \times 12 = 36$ and $2 \times 18 = 36$, we see directly that the LCM is 36.

(iv) We have to deal with 3, 5, and $9 = 3^2$. The LCM must contain at least two factors of 3 and one of 5. So the LCM is $5 \times 3^2 = 45$.

(v) 2, 4, 6. 2 divides 4, so we must have a factor of 4 in the LCM. Also, 4 and 6 both divide 12, but no smaller number and so the LCM is 12.

1.2.4 Manipulation of numbers

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Much of arithmetic is based on just a few operations: addition, subtraction, multiplication and division, satisfying a small number of rules. The extension of these rules to include **symbols** as well as numbers leads us on to **algebra** (Chapter 2).

Addition, denoted +, produces the **sum** of two numbers:

$$6 + 3 = 9 = 3 + 6 \quad \text{(addition is ‘commutative’)}$$

Subtraction, denoted −, produces the **difference** of two numbers:

$$6 - 3 = 3 = -(3 - 6) \quad \text{(minus sign changes signs in brackets)}$$

Multiplication, denoted by $a \times b$ or simply as ab in algebra, produces the **product** of two numbers:

$$6 \times 3 = (6)(3) = 18 = 3 \times 6 \quad \text{(multiplication is commutative)}$$

$a \cdot b$ is sometimes used to denote the product but can be confused with decimal notation in arithmetic.

Division, denoted by $a \div b$ or a/b or, better, $\dfrac{a}{b}$, produces the **quotient** of two numbers:

$$6 \div 3 = 6/3 = \tfrac{6}{3} = 2 \quad \text{(of course, } 6 \div 3 \neq 3 \div 6!\text{)}$$

Note that ÷ and / are very rarely used in written calculations, where we use the form $\frac{6}{3}$ unless we need to call into play ÷ or / because we have a large number of divisions. Also notice how we have simplified the quotient to 2. We always simplify such fractions to lowest form whenever we can (➤ 12).

The way the above and other arithmetic operations are combined is according to a set of conventional precedences – **the rules of arithmetic**. Thus we always perform multiplication before addition, so:

$$2 \times 3 + 5 = 6 + 5 = 11$$

Brackets can be used if we want to override such rules. For example:

$$2 \times (3 + 5) = 2 \times 8 = 16$$

In general, an arithmetic expression, containing numbers, ( ), x, $\div$, $+$, $-$, must be evaluated according to the following priorities:

BODMAS

$$\begin{array}{lll} \text{Brackets ( )} & & \text{first} \\ \left.\begin{array}{l}\text{Of (as in `fraction of' -- rarely used these days)} \\ \text{Division } \div \\ \text{Multiplication } \times\end{array}\right\} & & \text{second} \\ \left.\begin{array}{l}\text{Addition } + \\ \text{Subtraction } -\end{array}\right\} & & \text{third} \end{array}$$

If an expression contains only multiplication and division we work from left to right. If it contains only addition and subtraction we again work from left to right. If an expression contains powers or **indices** (Section 1.2.7) then these are evaluated after any brackets.

Products and quotients of negative numbers can be obtained using the following rules:

$$(+1)(+1) = +1 \quad (+1)(-1) = -1$$

$$(-1)(+1) = -1 \quad (-1)(-1) = +1$$

$$\frac{1}{(-1)} = -1$$

For example $(-2)(-3) = (-1)(-1)6 = 6$

Note that if you evaluate expressions on your calculator, it may not follow the BODMAS order, simply because of the way your calculator operates. However, BODMAS is the universal convention in Western mathematics and applies equally well to algebra, as we will see in Chapter 2.

Solution to review question 1.1.4

Following the BODMAS rule

(i) $2 + 3 - 7 = 5 - 7 = -2$

(ii) $4 \times 3 \div 2 = 12 \div 2 = 6$

(iii) $3 + 2 \times 5 = 3 + 10 = 13$

(iv) $(3 + 2) \times 5 = 5 \times 5 = 25$

(v) $3 + (2 \times 5) = 3 + 10 = 13$. In this case the brackets are actually unnecessary, since the BODMAS rules tell us to evaluate the multiplication first.

(vi) $18 \div 2 \times 3 = 9 \times 3 = 27$ following the convention of working from left to right.

(vii) $18 \div (2 \times 3) = 18 \div 6 = 3$ because the brackets override the left to right rule.

(viii) $-2-(4-5) = -2-(-1)$
$= -2+1$
$= -1$

(ix) $(4 \div (-2)) \times 3 - 4 = \left(\dfrac{4}{-2}\right) \times 3 - 4$
$= (-2) \times 3 - 4$
$= -6 - 4$
$= -10$

(x) $(3+7) \div 5 + (7-3) \times (2-4)$
$= 10 \div 5 + (4) \times (-2)$
$= 2 - 8$
$= -6$

Notice the care taken in these examples, spelling out each step. This may seem to be a bit laboured, but I would encourage you to take similar care, particularly when we come to algebra. Slips with brackets and signs crop up frequently in most people's calculations (mine included!). Whereas this may only lose you one or two marks in an exam, in real life, an error in sign can convert a stable control system into an unstable one, or a healthy bank balance into an overdraft.

1.2.5 Handling fractions

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A **fraction** or **rational number** is any quantity of the form

$$\frac{m}{n} \quad n \neq 0$$

where m, n are integers but n is not equal to 0. It is of course essential that $n \neq 0$, because as noted in Section 1.2.1 **division by zero is not defined**.

m is called the **numerator**

n is the **denominator**

If $m \geq n$ the fraction is said to be **improper**, and if $m < n$ it is **proper**.

A number expressed in the form $2\frac{1}{2}$ (meaning $2+\frac{1}{2}$) is called a **mixed fraction**. In mathematical expressions it is best to avoid this form altogether and write it as a **vulgar fraction**, $\frac{5}{2}$, instead, otherwise it might be mistaken for '$2 \times \frac{1}{2} = 1$', and it is also more difficult to do calculations such as multiplication and division using mixed fractions.

The numerator and denominator of a fraction may have common factors. These may be cancelled to reduce the fraction to its simplest or 'lowest' form:

$$\frac{6}{12} = \frac{2 \times 3}{3 \times 4} = \frac{2}{4} = \frac{1 \times 2}{2 \times 2} = \frac{1}{2}$$

Each of these forms are **equivalent fractions**, but clearly the last one is the simplest. However, sometimes one of the other forms may be convenient for particular purposes, such as adding fractions. A very common fraction where we tend **not** to cancel down in this way is the **percentage**. Thus we usually express 32/100 as ‘32 percent’ rather than as its equivalent, ‘8 out of 25’!

Fractions are multiplied ‘top by top and bottom by bottom’ as you might expect:

$$\frac{m}{n} \times \frac{p}{q} = \frac{mp}{nq} \quad (n, q \neq 0) \quad \text{e.g. } \frac{3}{2} \times \frac{5}{11} = \frac{15}{22}$$

with p and q also any integers. There may, of course, be common factors to cancel down, as for example in

$$\frac{3}{2} \times \frac{6}{7} = \frac{9}{7}$$

The inverse or **reciprocal** of a fraction is obtained by turning it upside down:

$$1\Big/\left(\frac{m}{n}\right) = \frac{n}{m} \quad \text{e.g. } 1\Big/\left(\frac{3}{2}\right) = \frac{2}{3}$$

where both m and n must be non-zero. So dividing by a vulgar fraction is done by inverting it and multiplying:

$$\left(\frac{p}{q}\right) \div \left(\frac{m}{n}\right) = \left(\frac{p}{q}\right)\Big/\left(\frac{m}{n}\right) = \frac{p}{q} \times \frac{n}{m} = \frac{np}{mq}$$

$$\text{e.g.} \qquad \frac{7}{2} \div \frac{14}{6} = \frac{7}{2} \times \frac{6}{14} = \frac{3}{2}$$

Multiplication and division of fractions are therefore quite simple. Addition and subtraction are less so.

Two fractions with the same denominator are easily added or subtracted:

$$\frac{m}{n} \pm \frac{p}{n} = \frac{m \pm p}{n} \quad \text{e.g. } \frac{3}{2} - \frac{1}{2} = \frac{3-1}{2} = \frac{2}{2} = 1$$

So to add and subtract fractions in general we rewrite them all with the same **common denominator**, which is the **lowest common multiple** of all the denominators. For example

$$\frac{3}{4} - \frac{4}{3} = \frac{3 \times 3}{12} - \frac{4 \times 4}{12} = \frac{9 - 16}{12} = -\frac{7}{12}$$

In the example, 12 is the LCM of 3 and 4.

An electrical example – resistances in parallel

Three resistances R_1, R_2, R_3 connected in parallel are equivalent to a single resistance R given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

So, for example if $R_1 = 2\Omega$, $R_2 = \frac{1}{2}\Omega$, $R_3 = \frac{3}{2}\Omega$ then

$$\frac{1}{R} = \frac{1}{2} + 2 + \frac{2}{3} \text{ (units of inverse ohms)}$$

or, with 6 the LCM of 2 and 3

$$= \frac{3}{6} + \frac{12}{6} + \frac{4}{6} = \frac{19}{6}\ (\Omega^{-1})$$

and so the equivalent resistance is

$$R = \frac{6}{19}\ \Omega$$

Finally, on fractions, recall the ideas of **ratio** and **proportion**. These are met early in our mathematical education, yet often continue to confuse us later in life. Specifically, it is not uncommon to see someone make errors such as:

$$\text{‘}\frac{a}{b} = \frac{1}{3} \text{ means } a = 1 \text{ and } b = 3\text{’}$$

so it is worth having a quick review of this topic.

The notation $a : b$ is used to indicate that the numbers a and b are in a certain ratio or proportionality to each other.

$$a : b = 1 : 3$$

simply means that

$$\frac{a}{b} = \frac{1}{3}$$

and this most certainly does not mean $a = 1$ and $b = 3$. For example

$$3 : 9 = 2 : 6 = 7 : 21 = 1 : 3$$

All $a : b = 1 : 3$ means is that

$$a = \frac{b}{3}$$

i.e. a is a third of b. If we are given a (or b) then we can find b (or a). The review question illustrates this.

In general, if we can write $a = kb$ where k is some given constant then we say ‘a is proportional to b’ and write this as $a \propto b$. a and b are then in the ratio $a : b = 1 : k$. On the other hand if we can write $a = k/b$ then we say ‘a is inversely proportional to b’ and write $a \propto 1/b$.

Solution to review question 1.1.5

A. (i) $\dfrac{4}{6} = \dfrac{2 \times 2}{2 \times 3} = \dfrac{2}{3}$

(ii) $\dfrac{18}{9} = \dfrac{9 \times 2}{9} = 2$

(iii) $\dfrac{\cancel{7}}{3} \times \dfrac{4}{\cancel{7}} = \dfrac{4}{3}$

(iv) $\dfrac{7}{5} \times \dfrac{3}{14} = \dfrac{1}{5} \times \dfrac{3}{2}$

(cancelling 7 from top and bottom, as in (iii))

$$= \frac{1 \times 3}{5 \times 2} = \frac{3}{10}$$

(v) $\dfrac{3}{4} \div \dfrac{4}{5} = \dfrac{3}{4} \times \dfrac{5}{4} = \dfrac{15}{16}$

(vi) $\dfrac{1}{2} + \dfrac{1}{3} = \dfrac{3}{2 \times 3} + \dfrac{2}{2 \times 3}$

on multiplying top and bottom appropriately to get the common denominator in both fractions,

$$= \frac{3}{6} + \frac{2}{6} = \frac{3+2}{6} = \frac{5}{6}$$

(vii) $\dfrac{1}{2} - \dfrac{1}{3} = \dfrac{3}{6} - \dfrac{2}{6}$

$$= \frac{3-2}{6} = \frac{1}{6}$$

(viii) $\dfrac{4}{15} - \dfrac{7}{3} = \dfrac{4}{15} - \dfrac{5 \times 7}{15}$

$$= \frac{4-35}{15}$$

$$= -\frac{31}{15}$$

(ix) $1 + \dfrac{1}{2} + \dfrac{1}{3} = \dfrac{6}{6} + \dfrac{3}{6} + \dfrac{2}{6}$

$$= \frac{11}{6}$$

Here we found the LCM of 2 and 3 (6) and put everything over this, including the 1.

(x) $\dfrac{2}{3} - \dfrac{3}{4} + \dfrac{1}{8}$

We want the LCM of 3, 4, 8. This is 24, so

$$\frac{2}{3} - \frac{3}{4} + \frac{1}{8} = \frac{2 \times 8}{24} - \frac{3 \times 6}{24} + \frac{3}{24}$$

$$= \frac{16 - 18 + 3}{24}$$

$$= \frac{1}{24}$$

B. If $a : b = 3 : 2$ then $\dfrac{a}{b} = \dfrac{3}{2}$ so $b = \dfrac{2a}{3}$. So, if $a = 6$ then $b = \dfrac{2 \times 6}{3} = 4$.

1.2.6 Factorial and combinatorial notation – permutations and combinations

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The factorial notation is a shorthand for a commonly-occurring expression involving positive integers. It provides some nice practice in manipulation of numbers and fractions, and gently introduces algebraic ideas. If n is some positive integer ≥ 1 then we write

$$n! = n(n-1)(n-2)\dots 2 \times 1$$

read as '***n* factorial**'. For example

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Notice that the factorial expression yields large values very quickly, that is $n!$ increases rapidly with n. In calculations involving factorials it is often useful to remember such results as

$$10! = 10 \times 9 \times 8 \times 7!$$

i.e. we can pick out a lower factorial if this is convenient, and this often helps with cancellations in expressions containing factorials.

Note that $1! = 1$. Also, while the above definition does not define 0!, the **convention** is adopted that

$$0! = 1$$

The factorial notation is useful in the binomial theorem (➤ 71) and in statistics. It can be used to count the number **permutations** of n objects, i.e. the number of ways of arranging n objects in a given order:

First object can be chosen in n ways
Second object can be chosen in $(n-1)$ ways
Third object can be chosen in $(n-2)$ ways
⋮
Last object can only be chosen in 1 way.

So the total number of permutations of n objects is

$$n \times (n-1) \times (n-2)\dots 2 \times 1 = n!$$

Note that $n! = n \times (n-1)!$

For 3 objects A, B, C, for example, there are $3! = 6$ permutations, which are:

ABC, ACB, BAC, BCA, CAB, CBA.

Each of these is the same **combination** of the objects A, B, C – that is a selection of three objects in which order is not important.

Now suppose we select just r objects from the n. Each such selection is a different combination of r objects from n. An obvious question is how many different permutations of r objects chosen from n can be formed in this way? This number is denoted by ${}^nP_r$. It may be evaluated by repeating the previous counting procedure, but only until we have chosen r objects:

The first may be chosen in n ways
The second may be chosen in $(n-1)$ ways
The third may be chosen in $(n-2)$ ways
$\vdots$
The rth may be chosen in $(n-(r-1))$ ways

So the total number of permutations will be

$$\begin{aligned} {}^nP_r &= n \times (n-1) \times (n-2) \times \ldots \times (n-r+1) \\ &= \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)(n-r-1)\ldots 2 \times 1}{(n-r)(n-r-1)\ldots 2 \times 1} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

For example the number of ways that we can permute 3 objects chosen from 5 distinct objects is

$${}^5P_3 = 5 \times 4 \times 3 = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = \frac{5!}{(5-3)!} = 60$$

Since the order does not matter in a **combination**, ${}^nP_r$ will include $r!$ permutations of the same combinations of r different objects. So the **number of combinations of r objects chosen from n** is

$$\frac{1}{r!}{}^nP_r = \frac{n!}{(n-r)!r!}$$

This is usually denoted by ${}^nC_r$ (called the '$n-C-r$' notation) or $\binom{n}{r}$ – 'choose r objects from n':

$${}^nC_r = \binom{n}{r} = \frac{n!}{(n-r)!r!} \quad \text{e.g. } {}^5C_3 = \binom{5}{3} = \frac{5!}{(5-3)!3!} = 10$$

which is very useful in binomial expansions (see Section 2.2.13) and other areas, simply as a notation, regardless of its 'counting' significance.

Solution to review question 1.1.6

A. (i) $3! = 3(3-1)(3-2) = 3 \times 2 \times 1 = 6$

(ii) $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

(iii) $\dfrac{24!}{23!} = \dfrac{24 \times 23!}{23!} = 24$

(iv) $\dfrac{12!}{9!3!} = \dfrac{12 \times 11 \times 10 \times 9!}{9!3!} = \dfrac{12 \times 11 \times 10}{6} = 220$

B. (i) (a) ${}^3C_2 = \dfrac{3!}{(3-2)!2!} = \dfrac{3!}{1!2!} = 3$

(b) ${}^6C_4 = \dfrac{6!}{(6-4)!4!} = \dfrac{6 \times 5 \times 4!}{2!4!} = 15$

(c) ${}^6P_3 = \dfrac{6!}{3!} = 6 \times 5 \times 4 = 120$

(ii) Two letters can be chosen from *ABCD* in

$$ {}^4C_2 = \frac{4!}{2!2!} = 6 \text{ ways} $$

(iii) There are $5! = 120$ permutations of five different letters.

1.2.7 Powers and indices

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Powers, or indices, provide, in the first instance, a shorthand notation for multiplying a number by itself a given number of times:

$$\begin{aligned} 2 \times 2 &= 2^2 \\ 2 \times 2 \times 2 &= 2^3 \\ 2 \times 2 \times 2 \times 2 &= 2^4 \end{aligned}$$

etc.

For a given number a we have

$$a^n = \underbrace{a \times a \times a \times \ldots \times a}_{n \text{ times}}$$

a is called the **base**, n the **power** or **index**. a^1 is simply a. By convention we take $a^0 = 1$ ($a \neq 0$). We introduce a^{-1} to denote the **reciprocal** $\dfrac{1}{a}$, since then $1 = a \times \dfrac{1}{a} = a^1 \times \dfrac{1}{a} = a^1 \times a^{-1} = a^{1-1} = a^0$ follows. In general, $a^{-n} = \dfrac{1}{a^n}$. From these definitions we can derive the **rules of indices**:

$$a^m \times a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

$$(ab)^n = a^n b^n$$

Note that for any index n, $1^n = 1$.

Examples

$$2^3 \times 2^4 = 2^{3+4} = 2^7$$

$$3^5 \times 3^0 = 3^5$$

$$(5^2)^3 = 5^6$$

$$(2^2)^0 = 2^0 = 1$$

$$4^3/2^2 = (2^2)^3/2^2 = 2^6/2^2 = 2^4$$

A **square root** of a positive number a, is any number that, when squared, yields the number a. We use $\sqrt{a}$ to denote the positive value of the square root (although the notation has to be stretched when we get to complex numbers). For example

$$2 = \sqrt{4} \quad \text{since } 2^2 = 4$$

Since $-2 = -\sqrt{4}$ also satisfies $(-2)^2 = 4$, $-\sqrt{4}$ is also a square root of 4. So the square roots of 4 are $\pm\sqrt{4} = \pm 2$.

We can similarly have cube roots of a number a, which yield a when they are cubed. If a is positive then $\sqrt[3]{a}$ denotes the positive value of the cube root. For example

$$2 = \sqrt[3]{8} \quad \text{because } 2^3 = 8$$

In the case of taking an odd root of a negative number the convention is to let $\sqrt{}$ denote the negative root value, as in $\sqrt[3]{-8} = -2$, for example.

The corresponding nth root of a number a is denoted in general by

$$\sqrt[n]{a} \quad \text{(also called a \textbf{radical})}$$

If n is even then a must be positive to yield a real root ($\sqrt{-1}$ is an **imaginary number**, forming the basis of complex numbers, see Chapter 12). In this case, because $(-1)^2 = 1$, there will be at least two values for the root differing only by sign. If n is odd then the nth root $\sqrt[n]{a}$ exists for both positive and negative values of a, as in $\sqrt[3]{-8} = -2$ above.

If a is a prime number such as 2, then $\sqrt{a}$ is an irrational number, i.e. it can't be expressed in rational form as a ratio of integers (6 ◀). This is not just a mathematical nicety. $\sqrt{2}$ for example, is the diagonal of the unit square, and yet because it is irrational, it can never be written down exactly as a rational number or fraction ($\sqrt{2} = 1.4142$ is, for example, only an approximation to $\sqrt{2}$ to four decimal places).

In terms of indices, roots are represented by fractional indices, for example:

$$\sqrt{a} = a^{\frac{1}{2}}$$

and in general

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

This fits in with the rules of indices, since

$$\left(a^{\frac{1}{n}}\right)^n = a^{\frac{1}{n}\times n} = a^1 = a$$

Fractional powers satisfy the same rules of indices as integer powers – but there are some new features:

- multiplicity of roots: $2^2 = (-2)^2 = 4$
- non-existence of certain roots of negative numbers: $\sqrt{-1}$ is not a real number
- irrational values for roots of primes and their multiples: $\sqrt{2}$ cannot be expressed as a fraction

Quantities such as $\sqrt{2}$, $\sqrt{3}, \ldots$ containing square roots of primes, are called **surds**. The term originates from the Greek word for mute, referring to a number that cannot 'speak' its value – because its decimal part never ends (see Section 1.2.8). In mathematical manipulation surds are always best left as they are – retaining the root sign. Any decimal form for them will simply be an approximation as noted for $\sqrt{2}$ above. Usually we try to manipulate surds so that the result is the simplest form, and none remain in denominators (although we would normally write, for example, $\sin 45^\circ = \frac{1}{\sqrt{2}}$). To do this we can use the rules of indices, and also a process known as **rationalisation**, in which surds in denominators are moved to the numerator. The ideas are illustrated in the solution to the review question.

Solution to review question 1.1.7

A. (i) $2^3 2^4 = 2^{3+4} = 2^7$ (leave it as a power, like this)

(ii) $\frac{3^4}{3^3} = 3^{4-3} = 3^1 = 3$

(iii) $(5^2)^3 = 5^{2\times 3} = 5^6$

(iv) $\frac{(3\times 4)^4}{(9\times 2^3)} = \frac{3^4 4^4}{9\times 2^3}$ (note both 3 and 4 are raised to the power 4)

Note that $9 = 3^2$, $4 = 2^2$, so we can write,

$$= \frac{3^4(2^2)^4}{3^2 2^3} = \frac{3^4 2^8}{3^2 2^3}$$

$$= 3^{4-2}2^{8-3}$$

$$= 3^2 2^5$$

(v) $\frac{16^2}{4^4} = \frac{(4^2)^2}{4^4} = \frac{4^4}{4^4} = 1$

(vi) $(-6)^2\left(-\frac{3}{2}\right)^3 = (-1)^2 6^2 (-1)^3 \frac{3^3}{2^3}$

$$= -\frac{6^2 3^3}{2^3}$$

on using $(-1)^2 = 1$, $(-1)^3 = -1$

$$= -\frac{(2\times 3)^2 3^3}{2^3} = -\frac{2^2 3^2 3^3}{2^3}$$

$$= -\frac{3^5}{2}$$

(vii) If it helps, just think of a and b as given numbers:

$$\frac{(-ab^2)^3}{a^2 b} = \frac{(-1)^3 a^3 b^6}{a^2 b}$$

$$= -ab^5$$

(viii) $2^2\left(\frac{1}{2}\right)^{-3} = 2^2(2^{-1})^{-3}$

$$= 2^2 2^3 = 2^5$$

The steps to watch out for in such problems are the handling of the minus signs and brackets, and dealing with the negative powers and reciprocals.

Don't forget that a^n, $n \geq 0$, is not defined for $a = 0$.

B. We can get a long way simply by using $\sqrt{ab} = \sqrt{a}\sqrt{b}$

(i) $\sqrt{50} = \sqrt{25 \times 2} = \sqrt{5^2}\sqrt{2} = 5\sqrt{2}$

(ii) $\sqrt{72} - \sqrt{8} = \sqrt{36 \times 2} - \sqrt{4 \times 2} = 6\sqrt{2} - 2\sqrt{2} = 4\sqrt{2}$

(iii) $(\sqrt{27})^3 = (3\sqrt{3})^3 = 3^3(\sqrt{3})^3 = 3^3 3\sqrt{3} = 3^4\sqrt{3} = 81\sqrt{3}$

(iv) $\left(\frac{\sqrt{2}\sqrt{3}}{4}\right)^2 = \left(\frac{\sqrt{3}}{2\sqrt{2}}\right)^2 = \frac{3}{4 \times 2} = \frac{3}{8}$

(v) $\frac{\sqrt{3}\sqrt{7}}{\sqrt{84}} = \frac{\sqrt{21}}{\sqrt{4 \times 21}} = \frac{\sqrt{21}}{2\sqrt{21}} = \frac{1}{2}$

(vi) To simplify $\frac{\sqrt{3} + 2\sqrt{2}}{\sqrt{3} - \sqrt{2}}$ we **rationalise** it by removing all surds from the denominator. To do this we use the algebraic identity:

$$(a - b)(a + b) \equiv a^2 - b^2$$

(see Section 2.2.1) and the removal of surds by squaring. For a $\sqrt{3} - \sqrt{2}$ on the bottom we multiply top and bottom by $\sqrt{3} + \sqrt{2}$, using:

$$(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2}) = (\sqrt{3})^2 - (\sqrt{2})^2 = 3 - 2 = 1$$

Thus:

$$\frac{\left(\sqrt{3}+2\sqrt{2}\right)}{\left(\sqrt{3}-\sqrt{2}\right)}\times\left(\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}\right)$$

$$=\frac{\left(\sqrt{3}+2\sqrt{2}\right)\left(\sqrt{3}+\sqrt{2}\right)}{\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}+\sqrt{2}\right)}$$

$$=\frac{\left(\sqrt{3}\right)^2+3\sqrt{2}\sqrt{3}+2\left(\sqrt{2}\right)^2}{\left(\sqrt{3}\right)^2-\left(\sqrt{2}\right)^2}$$

$$=\frac{3+3\sqrt{2}\sqrt{3}+4}{3-2}$$

$$=7+3\sqrt{6}$$

A similar ploy is used in the rationalisation or division of complex numbers (➤ 355)

(viii) $\left(\dfrac{3^{1/3}9^{1/3}}{27}\right)^2=\left(\dfrac{3^{1/3}3^{2/3}}{27}\right)^2=\left(\dfrac{3}{27}\right)^2=\left(\dfrac{1}{9}\right)^2=\dfrac{1}{81}$

This topic, powers and indices, often gives beginners a lot of trouble. If you are still the slightest bit unsure, go to the reinforcement exercises for more practice – there is no other way. This is, literally, power training!

1.2.8 Decimal notation

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You probably know that $\frac{1}{2}$ may be represented by the decimal 0.5, $\frac{1}{4}$ by 0.25 and so on. In fact any real number, a, $0 \le a < 1$, has a **decimal representation**, written

$$a = 0.d_1d_2d_3\ldots$$

where each d_i is one of the digits $0, 1, 2, \ldots, 9$, and the sequence may not terminate (see below). The term decimal actually refers to the **base** 10 and represents the fact that:

$$a = d_1 \times 10^{-1} + d_2 \times 10^{-2} + d_3 \times 10^{-3}\ldots$$

Note the importance of 'place value' here – the value of each of the digits depends on its place in the decimal.

Any real number can be represented by an integer part and such a decimal part. If, from some point on the decimal consists of a repeating string of one or more digits, then the decimal is said to be a **repeating** or **recurring** decimal. All rational numbers can be represented by a finite decimal representation or a recurring one. Irrational numbers cannot be represented in this way as a terminating or recurring decimal – thus the decimal representation of $\sqrt{2}$ is non-terminating:

$$\sqrt{2} = 1.4142135623\ldots$$

that is, the decimal part goes on forever.

All quantities measured in scientific or engineering experiments will have a finite decimal – every human observation of any kind is subject to a limited accuracy and so to a limited number of decimal places. Similarly any mechanical or electronic device can only yield a terminating decimal representation with a finite number of decimal places. In particular any number that you output on your calculator must represent a finite or recurring decimal – a rational number. So, for example no calculator or computer could ever yield the **exact** value of $\sqrt{2}$ or π. In practice even the most finicky engineer has limited need for decimal places – it can be shown that to measure the circumference of a circle girdling the known universe with an error no greater than the radius of a hydrogen atom requires the value of π to only 39 decimal places. π is actually known to many millions of decimal places. Nevertheless, irrational numbers such as $\sqrt{2}$, $\sqrt{3}$ actually occur frequently in engineering calculations, so we have to learn to handle them. $1/\sqrt{2}$ occurs for example in the rms value of an alternating current.

A useful way of expressing numerical value is by specifying a certain number of **significant digits**. To discuss these we need to be clear about zeros in numbers and what they represent. Some zeros are needed in a number simply as place holders – i.e. to tell us whether we are dealing with units, tens, hundreds, or tenths, hundredths, etc. For example in

$$15\mathbf{00},\ 0.\mathbf{00}23\underline{0},\ 2.103\underline{0}$$

the bold zeros are essential to hold place value – the only way to avoid them is to write the number in scientific notation (see below). The underlined zeros in these numbers are not strictly necessary and should only be included if they are significant – i.e. they represent a level of accuracy. For example if the number 1.24 is only accurate to the three 'significant figures' given then it could lie between 1.235 and 1.245. But if we write 1.240 then we are saying that there are four significant figures of accuracy and the number must lie between 1.2395 and 1.2405. The two end zeros in 1500 may or may not represent an accuracy to four figures – we have no way of knowing without further information. Therefore unless you are given further information, such zeros are assumed to be not significant. Similarly, the two first zeros in 0.002320 are assumed to be not significant – they are just place holders.

To count the number of **significant figures** in a number, start from the first non-zero digit on the left and count all digits (zero or not) to the right, counting final zeros if they are to the right of the decimal point, but not otherwise. Final zeros to the left of the decimal point are assumed not significant unless more information is given.

Examples

3.214 (4 sf), 2.041 (4 sf), 12.03500 (7 sf), 420 (2 sf), 0.003 (1 sf), 0.0801 (3 sf), 2.030 (4 sf), 500.00 (5 sf)

Sometimes numbers are approximated by terminating the digits after a given number of digits and replacing them with zeros. If this is done with no regard to the size of the removed digits, then we say the number has been 'chopped' or 'truncated'. For example 324829.1 chopped to 3 significant figures is 324000. Another, more accurate, method of approximation is 'rounding', in which we take account of the size of the removed digits.

When we 'round' a number we change the last non-zero digit not removed according to the size of the digits dropped. Specifically:

- If the digit to be removed is >5 then the immediately preceding digit is increased by 1
- If the digit to be removed is <5 the immediately preceding digit is left unchanged
- If the digit to be removed is equal to 5 then you may round up or down – one 'fair' way to do this is to round up if the previous digit is odd and down otherwise, for example.

Although 'chopping' may seem to give bigger errors because, for example, 324829.1 is closer to 325000 than 324000, it is usually the preferred method in computer arithmetic because it is much quicker than the more accurate 'rounding'.

Examples

213.457 chopped/rounded to 4 sf is 213.4/213.5, 56.0011 chopped/rounded to 4 sf is 56.00/56.00

We often need to convert between fractions and decimal representations. We can go from fraction to decimal by ordinary division. Conversely, we can convert a terminating decimal to the corresponding rational number by multiplying top and bottom by an appropriate factor as in, for example

$$0.625 = \frac{625}{1000} = \frac{25}{40} = \frac{5}{8}$$

Any decimal number can be written as a decimal number between 1 and 10 (the **mantissa**) multiplied by an appropriate power (the **exponent**) of 10. For example:

$$\begin{aligned} 74.932 &= 7.4932 \times 10 \\ \text{mantissa} &= 7.4932 \\ \text{exponent} &= 1 \end{aligned}$$

The purpose of such representation, called **scientific notation**, is to reduce very large and very small numbers to manageable form. For example

$$\begin{aligned} 573000000000000000 &= 5.73 \times 10^{17} \\ 0.0000000000000000000137 &= 1.37 \times 10^{-20} \end{aligned}$$

In engineering there is a variation on scientific notation that uses only multiples of 3 as exponents, i.e. as powers of 10. This is so that we can use the standard prefixes kilo, mega, micro, nano, etc.

Solution to review question 1.1.8

A. (i) (a) $\frac{1}{2} = 0.5$ (b) $-\frac{3}{2} = -1.5$ (c) $\frac{1}{3} = 0.\dot{3}$

where the dot above the final 3 denotes that this repeats forever: 0.3333... All the results in (a), (b), (c) are exact.

(d) By long division we find:

$$\tfrac{1}{7} = 0.142857142857\ldots = 0.\dot{1}4285\dot{7}$$

and the string of digits 142857 recur indefinitely as denoted by the over dots at the ends of the sequence. To six decimal places we can write

$$\tfrac{1}{7} \simeq 0.142857$$

B. (ii) (a) $0.3 = \dfrac{3}{10}$ (b) $0.67 = \dfrac{67}{100}$

(c) $0.\dot{6} = 0.6666\ldots = \tfrac{2}{3}$

To see this, let $x = 0.6666\ldots$ then $10x = 6.6666\ldots$ and subtracting gives

$$9x = 6$$

$$\text{so} \quad x = \tfrac{6}{9} = \tfrac{2}{3}$$

(d) $3.142 = \dfrac{3142}{1000}$

C. (i) $11.00132 = 1.100132 \times 10$
mantissa $= 1.100132$
exponent $= 1$

(ii) $1.56 = 1.56 \times 10^0$
mantissa $= 1.56$
exponent $= 0$

(iii) $203.45 = 2.0345 \times 10^2$
mantissa $= 2.0345$
exponent $= 2$

(iv) $0.0000321 = 3.21 \times 10^{-5}$
mantissa $= 3.21$
exponent $= -5$

D. To three significant figures we have

(i) $11.00132 = 11.0$

(ii) 1.56

(iii) $203.45 = 203$

(iv) $0.000321 = 0.000321$

1.2.9 Estimation

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With the availability of calculators we are now used to having enormous number crunching capability at our fingertips. But there are occasions when we don't have our hands on a calculator, or we need to get a rough order of magnitude check on a messy calculation. In such situations the engineer's most powerful tool has always been an ability to mentally estimate quantities and perform quick 'back of the envelope' (we still have them, despite email!) calculations. The trick is to approximate the numbers you are dealing with so that the calculations become simple, yet some sort of rough accuracy is retained. It is a matter of judgement and practice. Absolute values of numbers are less important than their relative values – for example 1021 is significant in

$$3 \times 1021 + 40 \times 234$$

but is relatively insignificant in

$$\frac{1021}{10} - 103372415$$

So, inspect all the numbers occurring in an expression and approximate them each to an appropriate order of magnitude, rounding as necessary, then perform the (hopefully) simplified calculation with the results.

Solution to review question 1.1.9

(i)
$$\frac{4.5 \times 10^5 \times 2.0012}{8.892 \times 10^4} \simeq \frac{4.5 \times 10^5 \times 2}{9 \times 10^4} \simeq \frac{10^5}{10^4} \simeq 10$$

So if your calculator gave you 101.2753 ... then you know you have slipped up on a decimal point.

(ii)
$$\frac{\sqrt{254 \times 10^4 + 28764.5}}{2.01 \times 10 - 2.54 \times 10^{-6}} \simeq \frac{\sqrt{254 \times 10^4 + 3 \times 10^4}}{2 \times 10}$$

neglecting 2.54×10^{-6} in comparison with 2.01×10

$$= \frac{\sqrt{257 \times 10^4}}{2 \times 10} \simeq \frac{\sqrt{256 \times 10^4}}{2 \times 10}$$
$$= \frac{\sqrt{16^2 \times 10^4}}{2 \times 10}$$

on replacing 257 by 256 for easy square rooting:

$$= \frac{16 \times 10^2}{2 \times 10} = 80$$

The answer to two decimal places is in fact 79.74.

1.3 Reinforcement

1.3.1 Types of numbers

◄◄2 5◄

A. Say what you can about the type and nature of the following numbers:

(i) 2
(ii) -3
(iii) 11
(iv) 21
(v) -0
(vi) $\frac{2}{3}$
(vii) $\frac{5}{2}$
(viii) $1\frac{2}{5}$
(ix) $-\frac{3}{7}$
(x) $\frac{18}{9}$
(xi) 0.0
(xii) 0.2
(xiii) -0.31
(xiv) 6.3
(xv) $\sqrt{3}$
(xvi) 3π
(xvii) e
(xviii) e^2
(xix) $-\sqrt{2}$
(xx) -1.371

(e is the base of natural logs – see Chapter 4)

B. Say all you can about the following expressions

(i) 0×3
(ii) $\frac{2}{0}$
(iii) $0 - 2$
(iv) $\frac{0}{-1}$
(v) $\frac{0+2}{0}$
(vi) 0^4
(vii) $0 \times \frac{1}{0}$
(viii) $-1 \times 0 + \frac{0}{2}$
(ix) $\frac{3 \times 0}{0}$
(x) 4^0
(xi) $0!$
(xii) $\frac{4!}{0!}$

1.3.2 Use of inequality signs

◄◄2 7◄

A. Using inequality signs, order all of the numbers in Q1.3.1A.

B. Suppose a, b and c are three non-zero positive numbers satisfying

$$a < b \leq c$$

What can you say about:

(i) a^2, b^2, c^2
(ii) $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$
(iii) $a + b, 2c$
(iv) $-a, -b, -c$
(v) $\sqrt{a}, \sqrt{b}, \sqrt{c}$?

1.3.3 Highest common factor and lowest common multiple

◀◀3 8◀

A. Express in terms of prime factors

(i) 2
(ii) -6
(iii) 21
(iv) 24
(v) -72
(vi) 81
(vii) $\frac{27}{14}$
(viii) 143
(ix) 391
(x) 205

B. Determine the highest common factor of each of the following sets of numbers

(i) 11, 88
(ii) 28, 40
(iii) 25, 1001
(iv) 20, 45, 90
(v) 14, 63, 95
(vi) 24, 72, 96
(vii) 36, 42, 54

C. Find the lowest common multiple of each of the following sets of numbers

(i) 2, 4
(ii) 5, 8
(iii) 12, 15
(iv) 6, 9, 27
(v) 12, 42, 60, 70
(vi) 66, 144

1.3.4 Manipulation of numbers

◀◀3 10◀

A. Evaluate

(i) $3-6\times 7$
(ii) $3(4-1)-2$
(iii) $(4-1)\div(6-3)$
(iv) $6-(3-6)\times 4$
(v) $24\div(6\div 2)$
(vi) $(24\div 6)\div 2$
(vii) $(2\times(3-1))\div(7-2(3-1))$

B. Evaluate

(i) $10-(2-3\times 4^2)$
(ii) $100-3(7-10)^3$
(iii) $-((-(-((-1)))))$
(iv) $-3(2-(3+1)(-4+2)+4\times 3)$
(v) $1-4(-2)$

C. Evaluate $4+5\times 2^3$ in its conventional meaning. Without these conventions how many pairs of brackets would be needed to make the meaning of the expression clear?

Now insert one pair of brackets in as many different non-trivial ways as possible and evaluate the resulting expressions (retaining the other usual conventions). Can any other results be obtained by the insertion of a second pair of brackets?

1.3.5 Handling fractions

◀◀3 12◀

A. Find in simplest form as a fraction

(i) $\frac{10}{12}$
(ii) $\frac{36}{9}$
(iii) $\frac{7}{3} \times \frac{2}{5}$
(iv) $\frac{14}{15} \div \frac{5}{7}$
(v) $\frac{1}{2} + \frac{1}{4}$
(vi) $\frac{1}{3} - \frac{1}{7}$
(vii) $\frac{21}{5} - \frac{7}{10}$
(viii) $1 - \frac{1}{2} + \frac{1}{3}$
(ix) $\frac{3}{5} - \frac{1}{2} + \frac{1}{6}$
(x) $\left(\frac{3}{7} - \frac{2}{9}\right) \div \left(\frac{1}{5} - \frac{1}{2}\right)$
(xi) $\frac{12}{1/2}$

B. (a) If $a : b = 7 : 3$ determine a for the following values of b: (i) 4, (ii) 3, (iii) -7, (iv) 15.
(b) Repeat for b with the same values for a.

C. If $a : b = 5 : 2$ evaluate as fractions

(i) $\frac{a}{b}$
(ii) $\frac{a}{a+b}$
(iii) $\frac{b}{a+b}$
(iv) $\frac{a-b}{a+b}$
(v) $\frac{a}{b} + \frac{b}{a}$
(vi) $\frac{a}{a+b} - \frac{b}{a-b}$
(vii) $\frac{a+b}{a-b} + \frac{a-b}{a+b}$

D. If a is proportional to b and $a = 6$ when $b = 4$, what are the values of the following

(i) a when $b = 3$
(ii) $\frac{a}{b} + \frac{b}{a}$
(iii) $\frac{a-b}{a+b}$
(iv) b when $a = 21$

1.3.6 Factorial and combinatorial notation – permutations and combinations

◀◀4 16◀

A. Evaluate

(i) $5!$
(ii) $10!$
(iii) $\frac{301!}{300!}$
(iv) $\frac{18!6!}{16!}$
(v) $\frac{14!}{(7!)^2 13}$
(vi) $\frac{10!}{4!6!}$
(vii) $10! + 11!$
(viii) $\frac{9!}{6!3!} - \frac{8!}{5!4!}$

B. Evaluate

(i) ${}^9C_2$
(ii) ${}^9C_7$
(iii) ${}^{11}C_4$
(iv) ${}^{10}C_4$
(v) ${}^{100}C_{100}$
(vi) ${}^7P_3$
(vii) ${}^6P_3{}^6C_3$

C. How many combinations of 4 different letters can be chosen from *ABCDEFG*?

1.3.7 Powers and indices

◀◀4 18◀

A. Evaluate in terms of powers of primes

(i) $2^2 2^3$ (ii) $3^4/3^2$ (iii) $6^3 \times 3^2/4$

(iv) $6^2 2^{-2} 3^2$ (v) $2^2 \times 4 \times 2^5$ (vi) $5^6/10^4$

(vii) $3^4 2^2 3^{-1}$ (viii) $49 \times 7/21^2$

B. Simplify (write as simplest products of powers of primes)

(i) $3^4 3^6 3^2$ (ii) $2^3 4^2/2^5$ (iii) $\dfrac{6^2 2^3 3^4 9}{2^2 3^3}$

(iv) $(4 \times 6)^6/(3^2 \times 4^2)$ (v) $27^5/9^5$ (vi) $(-4)^3/(-12)^4$

(vii) $\dfrac{3^2 4^6}{3^{-3} 2^{-1}}$ (viii) $\dfrac{8^{1/3} 27}{3^2 2^3}$

C. Show that the following are all the same number

$$\frac{2\sqrt{7}}{3\sqrt{5}},\quad \frac{\sqrt{28}}{\sqrt{45}},\quad \frac{14}{3\sqrt{35}},\quad \frac{2\sqrt{35}}{15},\quad \frac{2}{3}\sqrt{\frac{7}{5}},\quad \frac{2\sqrt{7}}{\sqrt{45}},\quad \frac{\sqrt{28}}{3\sqrt{5}},\quad \sqrt{\frac{28}{45}}$$

D. Express in terms of simplest surds

(i) $\sqrt{18}$ (ii) $\sqrt{20}$ (iii) $\sqrt{32}$ (iv) $\sqrt{52}$

(v) $\sqrt{512}$ (vi) $\sqrt{396}$ (vii) $\sqrt{108}$ (viii) $\sqrt{63}$

E. Rationalize

(i) $\dfrac{1}{\sqrt{7}}$ (ii) $-\dfrac{1}{\sqrt{3}}$ (iii) $\dfrac{1}{\sqrt{2}-1}$ (iv) $\dfrac{1}{4-\sqrt{10}}$

(v) $\dfrac{\sqrt{5}+1}{\sqrt{5}-1}$ (vi) $\dfrac{\sqrt{2}-2\sqrt{3}}{\sqrt{2}+\sqrt{3}}$ (vii) $\sqrt{\dfrac{1}{2}}+\sqrt{\dfrac{1}{4}}+\sqrt{\dfrac{1}{8}}$

(viii) $\sqrt{512}+\sqrt{128}+\sqrt{32}$

1.3.8 Decimal notation

◄◄4 22◄

A. Express in decimal form, to four decimal places in each case

(i) $-\frac{1}{2}$ (ii) $\frac{7}{2}$ (iii) $\frac{2}{3}$ (iv) $-\frac{2}{9}$

(v) $\frac{0}{6}$ (vi) $\frac{1}{8}$

B. Express as fractions

(i) 0.25 (ii) 0.125 (iii) 72.45 (iv) −0.312

(v) 0.17

C. Write the following numbers in scientific notation stating the mantissa and exponent

(i) 21.3241 (ii) 429.003 (iii) -0.000321 (iv) 0.00301
(v) 1,000,100 (vi) 300491.2

D. Write the numbers in **C** to (a) 3, (b) 6 significant figures.

1.3.9 Estimation

◄◄5 25◄

A. Assuming at $\pi \simeq 3.142$ and $e \simeq 2.718$ give approximate values for

(i) π^2 (ii) e^3 (iii) π^{-2} (iv) e^{-3}

B. Estimate the values of

(i) $\dfrac{0.0003 \times 3.1 \times 10^6}{9050}$ (ii) $\dfrac{2.01 + 403}{2.1 \times 10^{-3} - 29.9}$

(iii) $\dfrac{6 \times 10^5 + 1001e^3}{3.109 - 3.009}$ (iv) $\dfrac{3\pi^2 e^3}{63}$

1.4 Applications

1. An electrical circuit comprised of resistors only is illustrated in Figure 1.1

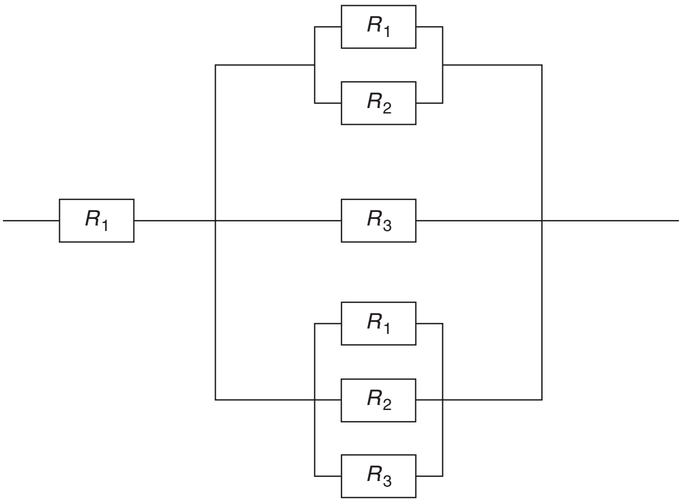

Figure 1.1

The equivalent resistance, R, of two resistors R_1 and R_2 in **series** is the **sum** of their resistances:

$$R = R_1 + R_2$$

The **reciprocal** of the equivalent resistance of two resistors in **parallel** is equal to the **sum of their reciprocals**:

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$$

(see Section 1.2.5)

(a) If, in the circuit of Figure 1.1, $R_1 = 1\Omega$, $R_2 = 2\Omega$, $R_3 = 3\Omega$ determine the overall equivalent resistance of the circuit, working entirely with fractions.
(b) Repeat the calculation using three decimal places accuracy and compare your result with (a).
(c) Suppose now that R_2 is not fixed, but can vary. Obtain the equivalent resistance in terms of R_2 using exact fractions.
(d) Using the result of (c), plot a graph of the equivalent resistance R for integer values of R_2 from 1 to 10.
(e) Discuss the calculations and results of (c), (d) in the light of the need to plot a useful graph.
(f) If we wish to increase slightly the equivalent resistance above that when $R_2 = 2\Omega$ what should we do with R_2, increase or decrease it?

2. Using the expressions for ${}^nC_r$ try a selection of values of n and r to investigate whether the following relations might be true.

(i) ${}^nC_r + {}^nC_{n-r} = {}^rC_n$ (ii) ${}^{n-1}C_r + {}^{n-1}C_{n-r} = {}^nC_r$
(iii) ${}^nC_r = {}^nC_{n-r}$

Prove any result that you suspect is true.

Answers to reinforcement exercises

1.3.1 Types of numbers

A. (i) 2 is a positive prime and even integer.

(ii) −3 is a negative integer.

(iii) 11 – integer, positive, prime, odd.

(iv) 21 – integer, positive, odd, composite (3 × 7).

(v) −0 – zero, both positive and negative.

(vi) $\frac{2}{3}$ – proper fraction, rational number, positive.

(vii) $\frac{5}{2}$ – improper fraction, positive, rational.

(viii) $1\frac{2}{5}$ is a positive mixed fraction expressible as the improper fraction $\frac{7}{2}$.

(ix) $-\frac{3}{7}$ – negative proper fraction.

(x) $\frac{18}{9}$ – improper fraction which can be cancelled down to lowest terms as an integer 2.

(xi) 0.0 – decimal representation, to one decimal place, of zero.

(xii) 0.2 is a decimal fraction expressible as the proper fraction $\frac{2}{10} = \frac{1}{5}$.

(xiii) -0.31 is a negative decimal fraction expressible as the negative proper fraction $-\frac{31}{100}$.

(xiv) 6.3 is a positive decimal number, expressible as the mixed fraction $6\frac{3}{10}$ or the improper fraction $\frac{63}{10}$.

(xv) $\sqrt{3}$ is a positive irrational number.

(xvi) 3π – irrational.

(xvii) e – irrational.

(xviii) e^2 – irrational.

(xix) $-\sqrt{2}$ is a negative, irrational number.

(xx) -1.371 is a negative decimal fraction expressible as the negative improper fraction $-\frac{1371}{1000}$.

B. (i) zero (ii) not defined (iii) negative integer
(iv) zero (v) not defined (vi) zero
(vii) not defined (viii) zero (ix) not defined
(x) $4^0 = 1$ (xi) $0! = 1$ (xii) 24

1.3.2 Use of inequality signs

A.
$$\begin{aligned} & 21 > 11 > 3\pi > e^2 > 6.3 > e > \frac{5}{2} > 2 \\ & = \frac{18}{9} > \sqrt{3} > 1\frac{2}{5} > \frac{2}{3} > 0.2 > 0.0 \\ & = -0 > -0.31 > -\frac{3}{7} > -\sqrt{2} > -1.371 > -3 \end{aligned}$$

B. (i) $a^2 < b^2 \leq c^2$ (ii) $\frac{1}{a} > \frac{1}{b} \geq \frac{1}{c}$ (iii) $a + b < 2c$
(iv) $-c \leq -b < -a$ (v) $\sqrt{a} < \sqrt{b} \leq \sqrt{c}$

1.3.3 Highest common factor and lowest common multiple

A. (i) 2 is already prime (ii) $-6 = -1 \times 2 \times 3$ (iii) $21 = 3 \times 7$

(iv) $24 = 2^3 \times 3$
(v) $-72 = -1 \times 2^3 \times 3^2$
(vi) $81 = 3^4$
(vii) $2^{-1} \times 3^3 \times 7^{-1}$
(viii) 11×13
(ix) 17×23
(x) 5×41

B. (i) 11
(ii) 4
(iii) 1
(iv) 5
(v) 1
(vi) 24
(vii) 6

C. (i) 4
(ii) 40
(iii) 60
(iv) 54
(v) 420
(vi) 1584

1.3.4 Manipulation of numbers

A. (i) -39
(ii) 7
(iii) 1
(iv) 18
(v) 8
(vi) 2
(vii) $\frac{4}{3}$

B. (i) 56
(ii) 181
(iii) 1
(iv) -66
(v) 9

C. 44; 2 pairs of brackets, $4 + (5 \times (2^3))$
With one pair we can get 72, 1004, 2744
With two pairs we could also get 18^3

1.3.5 Handling fractions

A. (i) $\frac{5}{6}$
(ii) 4
(iii) $\frac{14}{15}$
(iv) $\frac{98}{75}$
(v) $\frac{3}{4}$
(vi) $\frac{4}{21}$
(vii) $\frac{7}{2}$
(viii) $\frac{5}{6}$
(ix) $\frac{4}{15}$
(x) $-\frac{130}{189}$
(xi) 24

B. (a) (i) $\frac{28}{3}$
(ii) 7
(iii) $-\frac{49}{7}$
(iv) 35

(b) (i) $\frac{12}{7}$
(ii) $\frac{9}{7}$
(iii) -3
(iv) $\frac{45}{7}$

C. (i) $\frac{5}{2}$
(ii) $\frac{5}{7}$
(iii) $\frac{2}{7}$
(iv) $\frac{3}{7}$
(v) $\frac{29}{10}$
(vi) $\frac{1}{21}$
(vii) $\frac{58}{21}$

D. (i) $\frac{9}{2}$ (ii) $\frac{13}{6}$ (iii) $\frac{2}{5}$ (iv) 14

1.3.6 Factorial and combinatorial notation

A. (i) 120 (ii) 3628800 (iii) 301
(iv) 220320 (v) 264 (vi) 210
(vii) 43545600 (viii) 70

B. (i) 36 (ii) 36 (iii) 330
(iv) 210 (v) 1 (vi) 210
(vii) 2400

C. 35

1.3.7 Powers and indices

A. (i) 2^5 (ii) 3^2 (iii) 2×3^5 (iv) 3^4 (v) 2^9 (vi) $5^2 2^{-4}$
(vii) $2^2 3^3$ (viii) 7×3^{-2}

B. (i) 3^{12} (ii) 2^2 (iii) $2^3 3^5$
(iv) $2^{14} 3^4$ (v) 3^5 (vi) $-2^{-2} 3^{-4}$
(vii) $2^{13} 3^5$ (viii) $2^{-2} 3 = \frac{3}{4}$

D. (i) $3\sqrt{2}$ (ii) $2\sqrt{5}$ (iii) $2\sqrt{2}$
(iv) $2\sqrt{13}$ (v) $16\sqrt{2}$ (vi) $6\sqrt{11}$
(vii) $6\sqrt{3}$ (viii) $3\sqrt{7}$

E. (i) $\frac{\sqrt{7}}{7}$ (ii) $-\frac{\sqrt{3}}{3}$ (iii) $\sqrt{2} + 1$
(iv) $\frac{4 + \sqrt{10}}{6}$ (v) $\frac{3 + \sqrt{5}}{2}$ (vi) $3\sqrt{6} - 8$
(vii) $\frac{1}{4}(2 + 3\sqrt{2})$ (viii) $28\sqrt{2}$

1.3.8 Decimal notation

A. (i) -0.5000 (ii) 3.5000 (iii) 0.6667
(iv) -0.2222 (v) 0.0000 (vi) 0.1250

B. (i) $\frac{1}{4}$ (ii) $\frac{1}{8}$ (iii) $\frac{1449}{20}$

(iv) $-\dfrac{39}{125}$ (v) $\dfrac{17}{100}$

C.

	Given	Scientific Notation	Mantissa	Exponent
(i)	21.3241	2.13241×10	2.13241	1
(ii)	429.003	4.29003×10^2	4.29003	2
(iii)	−0.000321	-3.21×10^{-4}	−3.21	−4
(iv)	0.00301	3.01×10^{-3}	3.01	−3
(v)	1,000,100	1.0001×10^6	1.0001	6
(vi)	300491.2	3.004912×10^5	3.004912	5

D. (a) (i) 21.3 (ii) 429 (iii) −0.000321 (iv) 0.00301 (v) 1,000,000 (vi) 300,000

(b) (i) 21.3241 (ii) 429.003 (iii) −0.000321 (iv) 0.003010 (v) 1,000,100 (vi) 300491

1.3.9 Estimation

A. (i) 10 (ii) 20 (iii) $\dfrac{1}{10}$ (iv) $\dfrac{1}{20}$

B. (i) 0.1 (ii) −13 (iii) 62×10^5 (iv) 10